Recursive algorithm for evaluating vertex-type Feynman integrals

1992 J. Phys. A: Math. Gen. 25 5587

(http://iopscience.iop.org/0305-4470/25/21/017)

J. Phys. A: Math. Gen. 25 (1992) 5587–5596. Printed in the UK

Recursive algorithm for evaluating vertex-type Feynman integrals

A I Davydychev†‡

Instituut-Lorentz, University of Leiden, POB 9506, 2300 RA Leiden, The Netherlands

Received 30 April 1992

Abstract. An algorithm for evaluating vertex-type loop integrals is considered. It is based on applying the integration-by-parts technique. As an example, a class of massless integrals corresponding to triangle diagrams is considered. The presented method can also be applied to loop diagrams with larger number of external lines as well as to integrals with massive denominators.

1. Introduction

There are many important problems in contemporary elementary particle physics which require the development of effective methods, and algorithms, for evaluating Feynman loop diagrams. In particular, we mention the calculation of radiative corrections to various processes of elementary particle interaction, examination of Green function behaviour, studying coefficient functions in operator expansions, renormalization group analysis of β-functions, etc. Many appropriate references can be found, e.g. in reviews [1–4]. It should be noted that, up to the present time, the greatest success has been achieved in calculating various massless propagator-type loop diagrams. At the same time, the vertex-type diagrams (with three or more external lines) are also very important for studying many problems.

In realistic calculations we are often confronted with the necessity of evaluating Feynman integrals with different powers of denominators corresponding to the propagators of 'internal' particles. For example, such integrals occur in the following cases:

(i) when we deal with vector particles in covariant-type gauges (with the exception of the Feynman gauge);

(ii) when some of the external momenta of the diagram vanish;

(iii) when we differentiate the diagram with respect to the external momenta or masses (for example, when we use the sum rules method);

(iv) when we examine the compatibility of power-like solutions with loop equations for Green functions (e.g. when we study whether the $1/k^4$ infrared behaviour of the gluon propagator is consistent with the Schwinger–Dyson equations for the propagator and for the vertex; see also the review [5]);

(v) when we use the technique [6] to reduce tensor integrals to scalar ones; etc.

† E-mail address: DAVYD@COMPNET.NPIMSU.MSK.SU

‡ Present address: Institute for Nuclear Physics, Moscow State University, 119899, Moscow, Russia.

The present paper is devoted to examination of some problems of evaluating vertex-type integrals. As an example, we regard a class of vertex-type integrals corresponding to 'triangle' massless Feynman diagrams. In section 2 we present some results for these integrals obtained by use of the Mellin–Barnes contour integral representation and Feynman parameters. In section 3 we consider the recursive algorithm for calculating vertex-type integrals with different integer powers in the denominators. This algorithm is based on using the integration-by-parts technique [7]. In section 4 we formulate the main results and discuss the application of the method to more complicated integrals.

2. Some results for triangle massless diagrams

Let us consider the massless triangle diagram (see figure 1) with arbitrary external momenta p_1, p_2 and p_3 $(p_1 + p_2 + p_3 = 0)$. The corresponding Feynman integral is of the following form:

$$J(\nu_1, \nu_2, \nu_3) \equiv \int \frac{\mathrm{d}^n k}{((q_1 + k)^2)^{\nu_1}((q_2 + k)^2)^{\nu_2}((q_3 + k)^2)^{\nu_3}} \tag{2.1}$$

where n is the spacetime dimension (in the framework of dimensional regularization [8]), and ν_i $(i = 1,2,3)$ are the powers of denominators (or indices of the lines). As a rule, we shall put $n = 4 - 2\varepsilon$ $(\varepsilon \to 0)$. Nevertheless, the algorithm considered below can be applied to any values of n. In (2.1) it is understood that we use the 'causal' prescription for singularities in the pseudo-Euclidean space: $1/((q + k)^2)^\nu \leftrightarrow 1/((q + k)^2 + \mathrm{i}0)^\nu$. As a rule, we shall consider that the indices ν_i are integers. We also note that integrals (2.1) are symmetric with respect to the permutations of $(p_1, \nu_1), (p_2, \nu_2), (p_3, \nu_3)$.

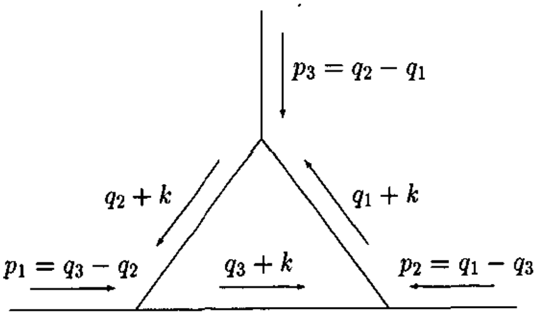

Figure 1. The arrangement of momenta in a triangle diagram.

If one of the indices ν_i vanishes then the integral (2.1) can be expressed through the well known one-loop two-point integrals $I(\nu, \nu'|p)$ (p is the external momentum):

$$\begin{aligned} J(\nu_1, \nu_2, 0) &= I(\nu_1, \nu_2|p_3) = \pi^{n/2} \mathrm{i}^{1-n} G(\nu_1, \nu_2)(p_3^2)^{n/2-\nu_1-\nu_2} \\ J(\nu_1, 0, \nu_3) &= I(\nu_1, \nu_3|p_2) = \pi^{n/2} \mathrm{i}^{1-n} G(\nu_1, \nu_3)(p_2^2)^{n/2-\nu_1-\nu_3} \\ J(0, \nu_2, \nu_3) &= I(\nu_2, \nu_3|p_1) = \pi^{n/2} \mathrm{i}^{1-n} G(\nu_2, \nu_3)(p_1^2)^{n/2-\nu_2-\nu_3} \end{aligned} \tag{2.2}$$

where

$$G(\nu,\nu') = G(\nu',\nu) \equiv \frac{\Gamma(n/2-\nu)\Gamma(n/2-\nu')\Gamma(\nu+\nu'-n/2)}{\Gamma(\nu)\Gamma(\nu')\Gamma(n-\nu-\nu')}. \tag{2.3}$$

It can also be noted that if one of the indices ν_1, ν_2, ν_3 is negative then this integral also can be reduced to two-point results (2.2). Moreover, if two or three indices ν_i are non-positive integers then these integrals correspond to 'tadpole' diagrams and are equal to zero (in the framework of dimensional regularization; see also the review [9]). Thus, it is a most interesting problem to study the region where all ν_i are positive. In this case we shall consider the results (2.2) as 'boundary' integrals.

In contrast to propagator-type integrals (2.2) with a simple power-like momentum behaviour, in general the integrals (2.1) depend on three momentum invariants p_1^2, p_2^2 and p_3^2. One can construct from these momenta squared two dimensionless variables, for example:

$$x \equiv \frac{p_1^2}{p_3^2} \qquad \text{and} \qquad y \equiv \frac{p_2^2}{p_3^2}. \tag{2.4}$$

Here we are compelled to use one of the invariants, p_3^2 , as a dimensionless-making parameter since we have no other massive parameters in the massless case. For arbitrary values of ν_1, ν_2, ν_3 and n one can derive the following two-fold Mellin–Barnes representation [10]:

$$\begin{aligned} J(\nu_1,\nu_2,\nu_3) = {} & \frac{\pi^{n/2} \mathrm{i}^{1-n} (p_3^2)^{n/2-\nu_1-\nu_2-\nu_3}}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)\Gamma(n-\nu_1-\nu_2-\nu_3)} \frac{1}{(2\pi \mathrm{i})^2} \\ & \times \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \mathrm{d}s\, \mathrm{d}t\, x^s y^t \Gamma(-s)\Gamma(-t)\Gamma(n/2-\nu_2-\nu_3-s) \\ & \times \Gamma(n/2-\nu_1-\nu_3-t)\Gamma(\nu_3+s+t)\Gamma(\nu_1+\nu_2+\nu_3-n/2+s+t) \end{aligned} \tag{2.5}$$

where the integration contours separate the 'right' and 'left' series of poles of gamma functions in the integrand (see e.g. [11]). It can be noted that an analogous representation for integrals corresponding to diagrams with an arbitrary number of external lines has been presented in [12].

Closing the s and t contours in (2.5) to the right yields [13]:

$$\begin{aligned} J(\nu_1,\nu_2,\nu_3) = {} & \frac{\pi^{n/2} \mathrm{i}^{1-n} (p_3^2)^{n/2-\nu_1-\nu_2-\nu_3}}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)\Gamma(n-\nu_1-\nu_2-\nu_3)} \\ & \times \{\Gamma(\nu_3)\Gamma(\nu_1+\nu_2+\nu_3-n/2)\Gamma(n/2-\nu_1-\nu_3)\Gamma(n/2-\nu_2-\nu_3) \\ & \times F_4(\nu_3, \nu_1+\nu_2+\nu_3-n/2; \\ & \qquad \nu_2+\nu_3-n/2+1, \nu_1+\nu_3-n/2+1 | x, y) \\ & + y^{n/2-\nu_1-\nu_3}\Gamma(\nu_2)\Gamma(n/2-\nu_1)\Gamma(\nu_1+\nu_3-n/2)\Gamma(n/2-\nu_2-\nu_3) \\ & \times F_4(\nu_2, n/2-\nu_1; \nu_2+\nu_3-n/2+1, n/2-\nu_1-\nu_3+1 | x, y) \end{aligned}$$

$$+ x^{n/2-\nu_2-\nu_3}\Gamma(\nu_1)\Gamma(n/2-\nu_2)\Gamma(n/2-\nu_1-\nu_3)\Gamma(\nu_2+\nu_3-n/2)$$
$$\times F_4(\nu_1, n/2-\nu_2; n/2-\nu_2-\nu_3+1, \nu_1+\nu_3-n/2+1|x,y)$$
$$+ x^{n/2-\nu_2-\nu_3}y^{n/2-\nu_1-\nu_3}$$
$$\times \Gamma(n-\nu_1-\nu_2-\nu_3)\Gamma(n/2-\nu_3)\Gamma(\nu_2+\nu_3-n/2)\Gamma(\nu_1+\nu_3-n/2)$$
$$\times F_4(n-\nu_1-\nu_2-\nu_3, n/2-\nu_3;$$
$$n/2-\nu_2-\nu_3+1, n/2-\nu_1-\nu_3+1|x,y)\} \qquad (2.6)$$

where

$$F_4(a,b;c,d|x,y) = \sum_{j=0}^{\infty}\sum_{l=0}^{\infty} \frac{x^j y^l}{j!l!} \frac{(a)_{j+l}(b)_{j+l}}{(c)_j (d)_l} \qquad (2.7)$$

is the Appell's hypergeometric function of two variables (see, e.g. [11], [14]), and $(a)_j \equiv \Gamma(a+j)/\Gamma(a)$ denotes the Pochhammer symbol. If ν_3, ν_2 or ν_1 vanishes we obtain, from (2.6), the results (2.2) while at $\nu_1+\nu_2+\nu_3 = n$ we get the uniqueness condition (see, e.g. in [15]). Using the representation (2.5) we can also obtain results in terms of other dimensionless combinations of momenta.

Let us examine the important special case, $\nu_1 = \nu_2 = \nu_3 = 1$. Then we can consider the limit $n \to 4 (\varepsilon \to 0)$, and we get

$$J(1,1,1)|_{n=4} = \frac{\mathrm{i}\pi^2}{p_3^2}\Phi(x,y) \qquad (2.8)$$

with

$$\Phi(x,y) \equiv \left(\frac{\pi^2}{3} + \ln x \ln y\right) F_4(1,1;1,1|x,y)$$
$$+ 2\ln x(\partial_a F_4 + \partial_d F_4) + 2\ln y(\partial_a F_4 + \partial_c F_4)$$
$$+ 2(\partial_a^2 F_4 + \partial_a \partial_b F_4 + 2\partial_a \partial_c F_4 + 2\partial_a \partial_d F_4 + 2\partial_c \partial_d F_4). \qquad (2.9)$$

Here we introduced notation for the derivatives of the function (2.7) with respect to the parameters a, b, c and d (taking into account the symmetry of (2.7) with respect to a and b), for example:

$$\partial_a F_4 \equiv \left(\frac{\partial}{\partial a} F_4(a,b;c,d|x,y)\right)\bigg|_{a=b=c=d=1}$$

etc. The coefficients of the parametric derivative expansions in x and y can be easily obtained by differentiation of the coefficients of (2.7), and they contain ψ-functions and their derivatives. Thus, the formula (2.8) gives us the asymptotic expansion for small values of x and y (with due regard for $\ln x, \ln y$ and $\ln x \ln y$ terms).

To pass to the standard representation of the result for $J(1,1,1)$ (see e.g. [16]), it is convenient to use the reduction formulae for the function F_4 at special values of parameters (see, e.g., [11, p 102] or [17, p 453]). Using these formulae and introducing the notation

$$\lambda(x,y) \equiv \sqrt{(1-x-y)^2 - 4xy} \qquad (2.10)$$

we find that

$$F_4(1,1;1,1|x,y) = \frac{1}{\lambda(x,y)}$$

$$\partial_a F_4 + \partial_c F_4 = \frac{1}{\lambda} \ln\left(\frac{1+x-y-\lambda}{2x}\right)$$

$$\partial_a F_4 + \partial_d F_4 = \frac{1}{\lambda} \ln\left(\frac{1-x+y-\lambda}{2y}\right)$$

$$\begin{aligned}\partial_a^2 F_4 + \partial_a\partial_b F_4 + 2\partial_a\partial_c F_4 + 2\partial_a\partial_d F_4 + 2\partial_c\partial_d F_4 \\ = \frac{1}{\lambda}\left\{\ln\left(\frac{1+x-y-\lambda}{2x}\right)\ln\left(\frac{1-x+y-\lambda}{2y}\right)\right. \\ \left. -\mathrm{Li}_2\left(\frac{1+x-y-\lambda}{2}\right) - \mathrm{Li}_2\left(\frac{1-x+y-\lambda}{2}\right)\right\}\end{aligned}$$

where $\mathrm{Li}_2(z)$ is the Euler's dilogarithm. Taking into account these conditions we get

$$\begin{aligned}\Phi(x,y) = \frac{1}{\lambda}\left\{2\ln\left(\frac{1+x-y-\lambda}{2}\right)\ln\left(\frac{1-x+y-\lambda}{2}\right) - \ln x \ln y\right. \\ \left. -2\mathrm{Li}_2\left(\frac{1+x-y-\lambda}{2}\right) - 2\mathrm{Li}_2\left(\frac{1-x+y-\lambda}{2}\right) + \frac{\pi^2}{3}\right\}\end{aligned} \quad (2.11)$$

where $\lambda = \lambda(x,y)$ is defined by the formula (2.10). The formula (2.11) gives us the standard representation for the integral (2.8). The results of such type are well known (see e.g. [16]). It should be noted that the same result (2.11) can be obtained by using the Feynman parametric representation,

$$\Phi(x,y) = \int_0^1 \mathrm{d}\xi \frac{\ln((1-\xi)x + \xi y) - \ln\xi - \ln(1-\xi)}{((1-\xi)x + \xi y - \xi(1-\xi))} \quad (2.12)$$

or by the dispersion technique (see e.g. [18]).

In our opinion, representations of the type of (2.6) and (2.9) are more useful for studying the asymptotic behaviour of the results, while in numerical calculations it is more convenient to use results of the type of (2.11).

3. Recurrence relations and the algorithm

Let us turn to examining integrals (2.1) with other positive powers of denominators ν_1, ν_2 and ν_3. To do this we could use either the general result (2.6) or parametric integral representations of the type of (2.12). However, both these ways are rather labour-consuming. In this section we shall consider the recursive algorithm for evaluating integrals with different powers of denominators which is based on the integration by parts technique [7]. Note that a similar procedure for the propagator-

$$\lambda(x,y) \equiv \sqrt{(1-x-y)^2 - 4xy} \quad (2.10)$$

We use the property [7] that the dimensionally regularized integrals with full divergence in the integrand vanish,

$$\int d^n k \frac{\partial}{\partial k_\mu} \left\{ \frac{(q_i + k)_\mu}{((q_1 + k)^2)^{\nu_1}((q_2 + k)^2)^{\nu_2}((q_3 + k)^2)^{\nu_3}} \right\} = 0 \qquad i = 1,2,3. \quad (3.1)$$

Hence we get the following relations for the integrals (2.1):

$$\begin{aligned}
&\nu_2 p_3^2 J(\nu_1, \nu_2 + 1, \nu_3) + \nu_3 p_2^2 J(\nu_1, \nu_2, \nu_3 + 1) \\
&\qquad = (2\nu_1 + \nu_2 + \nu_3 - n) J(\nu_1, \nu_2, \nu_3) \\
&\qquad\quad + \nu_2 J(\nu_1 - 1, \nu_2 + 1, \nu_3) + \nu_3 J(\nu_1 - 1, \nu_2, \nu_3 + 1) \\
&\nu_1 p_3^2 J(\nu_1 + 1, \nu_2, \nu_3) + \nu_3 p_1^2 J(\nu_1 + 1, \nu_2, \nu_3) \\
&\qquad = (\nu_1 + 2\nu_2 + \nu_3 - n) J(\nu_1, \nu_2, \nu_3) \qquad (3.2) \\
&\qquad\quad + \nu_1 J(\nu_1 + 1, \nu_2 - 1, \nu_3) + \nu_3 J(\nu_1, \nu_2 - 1, \nu_3 + 1) \\
&\nu_1 p_2^2 J(\nu_1 + 1, \nu_2, \nu_3) + \nu_2 p_1^2 J(\nu_1, \nu_2 + 1, \nu_3) \\
&\qquad = (\nu_1 + \nu_2 + 2\nu_3 - n) J(\nu_1, \nu_2, \nu_3) \\
&\qquad\quad + \nu_1 J(\nu_1 + 1, \nu_2, \nu_3 - 1) + \nu_2 J(\nu_1, \nu_2 + 1, \nu_3 - 1).
\end{aligned}$$

We have written the equations (3.2) in such a way that on the right-hand sides we have integrals with the sum of the indices $\sigma = \nu_1 + \nu_2 + \nu_3$, while on the left-hand sides we have integrals with $\sigma = \nu_1 + \nu_2 + \nu_3 + 1$. Thus, we can regard (3.2) as a system of simultaneous equations with respect to the integrals $J(\nu_1+1, \nu_2, \nu_3)$, $J(\nu_1, \nu_2+1, \nu_3)$ and $J(\nu_1, \nu_2, \nu_3 + 1)$ with the determinant

$$\Delta = \begin{vmatrix} 0 & \nu_2 p_3^2 & \nu_3 p_2^2 \\ \nu_1 p_3^2 & 0 & \nu_3 p_1^2 \\ \nu_1 p_2^2 & \nu_2 p_1^2 & 0 \end{vmatrix} = 2\nu_1\nu_2\nu_3 p_1^2 p_2^2 p_3^2. \qquad (3.3)$$

Solving this system we obtain, e.g., that

$$\begin{aligned}
J(\nu_1, \nu_2, \nu_3 + 1) = \frac{1}{2\nu_3 p_1^2 p_2^2} \{ &((2\nu_1 + \nu_2 + \nu_3 - n) p_1^2 \\
&+ (\nu_1 + 2\nu_2 + \nu_3 - n) p_2^2 - (\nu_1 + \nu_2 + 2\nu_3 - n) p_3^2) J(\nu_1, \nu_2, \nu_3) \\
&+ \nu_2 p_1^2 J(\nu_1 - 1, \nu_2 + 1, \nu_3) + \nu_3 p_1^2 J(\nu_1 - 1, \nu_2, \nu_3 + 1) \\
&+ \nu_1 p_2^2 J(\nu_1 + 1, \nu_2 - 1, \nu_3) + \nu_3 p_2^2 J(\nu_1, \nu_2 - 1, \nu_3 + 1) \\
&- \nu_1 p_3^2 J(\nu_1 + 1, \nu_2, \nu_3 - 1) - \nu_2 p_3^2 J(\nu_1, \nu_2 + 1, \nu_3 - 1) \} \qquad (3.4)
\end{aligned}$$

and also analogous results for $J(\nu_1 + 1, \nu_2, \nu_3)$ and $J(\nu_1, \nu_2 + 1, \nu_3)$. Using these formulae we define three integrals on the plane $\sigma = \nu_1 + \nu_2 + \nu_3 + 1$ through the integral $J(\nu_1, \nu_2, \nu_3)$ and six contiguous integrals on the plane $\sigma = \nu_1 + \nu_2 + \nu_3$. One can easily see that, applying these formulae consecutively an appropriate number of times, we can express the integral with any positive integer ν_1, ν_2, ν_3 in terms of $J(1, 1, 1)$ and boundary integrals (2.2).

One can obviously imagine this process by use of the (ν_1, ν_2, ν_3) coordinate space, considering the planes $\nu_1 + \nu_2 + \nu_3 = \sigma$, σ being a positive integer. We are interested in the region where $\nu_i \geqslant 0$ $(i = 1, 2, 3)$. The cases when at least one of the ν_i vanishes are trivial (2.2). The first plane containing the integral with all ν_i being positive corresponds to the case $\sigma = 3$. The corresponding integral $J(1,1,1)$ (at $n = 4$) was calculated in the previous section. The second plane ($\sigma = 4$) involves three integrals with positive ν_i : $J(1,1,2), J(1,2,1)$ and $J(2,1,1)$. Using the relation (3.4) at $\nu_1 = \nu_2 = \nu_3 = 1$ we get

$$J(1,1,2) = \frac{1}{p_1^2 p_2^2}\{(p_1^2 + p_2^2 - p_3^2)\varepsilon J(1,1,1) + p_1^2 I(2,1|p_1) + p_2^2 I(2,1|p_2) - p_3^2 I(2,1|p_3)\} \tag{3.5}$$

with $\varepsilon = (4-n)/2$. It should be noted that the term $\varepsilon J(1,1,1)$ disappears as $\varepsilon \to 0$ from the formula (3.5) since the integral $J(1,1,1)$ is finite as $n \to 4$ (2.8), (2.9), (2.11). Thus, in the case $n \to 4 J(1,1,2)$ can be expressed through boundary propagator-type integrals (2.2) only. Using the expansion

$$(p^2)^{-\varepsilon} = 1 - \varepsilon \ln p^2 + \mathcal{O}(\varepsilon^2) \tag{3.6}$$

and the formula (2.3) we obtain (keeping the singular and finite as $\varepsilon \to 0$ terms only)

$$J(1,1,2) = A\frac{1}{p_1^2 p_2^2}\left\{-\frac{1}{\varepsilon} + \ln p_1^2 + \ln p_2^2 - \ln p_3^2\right\} \tag{3.7}$$

with

$$A = \mathrm{i}^{1+2\varepsilon}\pi^{2-\varepsilon}\frac{\Gamma^2(1-\varepsilon)\Gamma(1+\varepsilon)}{\Gamma(1-2\varepsilon)} = \mathrm{i}^{1+2\varepsilon}\pi^{2-\varepsilon}\Gamma(1+\varepsilon) + \mathcal{O}(\varepsilon^2) \tag{3.8}$$

(this is a common factor for all integrals). Note that the $1/\varepsilon$ pole in (3.7) has infrared origin, due to the second power of massless denominator. It is understood in the formulae (3.6) and (3.7) that the arguments of logarithms (p_i^2) are made dimensionless either by massive parameters of dimensional regularization [8] or by one of the p_i^2 (for example, by p_3^2). In this case we find(2.4)

$$J(1,1,2) = A(p_3^2)^{-2-\varepsilon}\frac{1}{xy}\left\{-\frac{1}{\varepsilon} + \ln x + \ln y\right\}. \tag{3.9}$$

The same result (3.9) can be also obtained from the general formula (2.6) (however, this requires a considerably more cumbersome calculation). In this section we shall prefer the form (3.7) rather than (3.9) to keep explicit symmetry of the results.

The expressions for remaining integrals, $J(1,2,1)$ and $J(2,1,1)$, can be obtained from (3.7) by using the symmetry properties:

$$J(1,2,1) = A\frac{1}{p_1^2 p_3^2}\left\{-\frac{1}{\varepsilon} + \ln p_1^2 - \ln p_2^2 + \ln p_3^2\right\} \tag{3.10}$$

$$J(2,1,1) = A\frac{1}{p_2^2 p_3^2}\left\{-\frac{1}{\varepsilon} - \ln p_1^2 + \ln p_2^2 + \ln p_3^2\right\}. \tag{3.11}$$

Note that the same results (3.10) and (3.11) can also be obtained by using other recurrence formulae of the type of (3.4). Thus, all the integrals with $\sigma = \nu_1 + \nu_2 + \nu_3 = 4$ can be expressed as $\varepsilon \to 0$ in terms of propagator-type integrals (2.2) and do not contain complicated functions of the type (2.11).

Moreover, from (3.4) it is clear that, calculating any other integrals with $\sigma > 4$, we can always express them through integrals with $\sigma = 4$ and boundary integrals (2.2). Note that recurrence relations (3.4) cannot give us the coefficients which are singular in ε. Therefore, the integral $J(1,1,1)$ will always enter with the factor ε, and it will disappear from the formulae as $\varepsilon \to 0$.

Thus, we have proved that for any integer values of ν_1, ν_2, ν_3 (with the exception of the case $\nu_1 = \nu_2 = \nu_3 = 1$) the integrals $J(\nu_1, \nu_2, \nu_3)$ (2.1) can be expressed as $n \to 4$ ($\varepsilon \to 0$) in terms of linear combinations of boundary integrals (2.2) with regular (in ε) coefficients. Therefore, all such integrals contain powers and logarithms of external momenta squared only. The only 'complicated' integral $J(1,1,1)$ is defined by the formulae (2.8), (2.9), (2.11). It can be noted that an analogous situation occurred earlier, when evaluating axial-gauge propagator-type integrals (see, e.g., [21]).

Using recurrence relations (3.4) can be easily algorithmized. To do this, we have used the REDUCE system [22]. The results for some other integrals are presented in the appendix.

4. Conclusion

Thus, using triangle massless diagrams as an example, in the present paper we have examined some problems of evaluating vertex-type Feynman integrals. In section 2 the general results (2.5) and (2.6) for integrals (2.1) were considered, and the expressions (2.8), (2.9) and (2.11) for the integral $J(1,1,1)$ (at $n = 4$) were presented. In section 3 we examined a recursive method of evaluating integrals (2.1) with other positive powers of denominators ν_i which was based on the integration-by-parts technique [7]. It was proved that as $n \to 4$ the integrals with any positive values of ν_i could be reduced to two-point integrals (2.2), with the exception of the case $\nu_1 = \nu_2 = \nu_3 = 1$. The main recursion formula (3.4) is true for any n; therefore, the presented algorithm can be applied to integrals with any values of the spacetime dimension.

It should be noted that the presented technique can also be applied to vertex-type integrals with larger numbers of external lines N ($N > 3$). In this case we obtain a system of N equations, instead of (3.2).

Finally, we note that the examined method can also be used to evaluate massive vertex-type Feynman integrals. For example, if we substitute in the integral (2.1) the massless denominators $(q_i + k)^2$ by the massive ones, $((q_i + k)^2 - m_i^2)$, then the right-hand sides of the equations (3.2) will not change while the determinant (3.3) will be of the following form:

$$\Delta = \begin{vmatrix} -2\nu_1 m_1^2 & \nu_2(p_3^2 - m_1^2 - m_2^2) & \nu_3(p_2^2 - m_1^2 - m_3^2) \\ \nu_1(p_3^2 - m_1^2 - m_2^2) & -2\nu_2 m_2^2 & \nu_3(p_1^2 - m_2^2 - m_3^2) \\ \nu_1(p_2^2 - m_1^2 - m_3^2) & \nu_2(p_1^2 - m_2^2 - m_3^2) & -2\nu_3 m_3^2 \end{vmatrix}. \qquad (4.1)$$

As a result, the solutions of the type of (3.4) become more cumbersome. Nevertheless, the main features of the algorithm are also the same for the massive case.

Acknowledgments

I would like to thank Instituut-Lorentz for their hospitality. I enjoyed useful discussions with Dr F A Berends, W L van Neerven and J B Tausk. I also wish to thank Dr E E Boos for fruitful discussions, and Dr A I Alekseev for making me familiar with the paper [19].

Appendix

Here we present the results for some other integrals (2.1) obtained by using the recurrence relations (3.4):

$$J(1,2,2) = A\frac{1}{(p_1^2)^2 p_2^2 p_3^2}\left\{-(p_2^2+p_3^2)\frac{1}{\varepsilon} + p_1^2 - p_2^2 - p_3^2 \right.$$
$$\left. + (p_2^2+p_3^2)\ln p_1^2 - (p_2^2 - p_3^2)(\ln p_2^2 - \ln p_3^2)\right\} \qquad \text{(A1)}$$

$$J(1,1,3) = A\frac{1}{2(p_1^2)^2(p_2^2)^2}\left\{-(p_1^2+p_2^2-p_3^2)\frac{1}{\varepsilon} - 3p_1^2 - 3p_2^2 + p_3^2 \right.$$
$$\left. + (p_1^2+p_2^2-p_3^2)(\ln p_1^2 + \ln p_2^2 - \ln p_3^2)\right\} \qquad \text{(A2)}$$

$$J(2,2,2) = A\frac{1}{(p_1^2)^2(p_2^2)^2(p_3^2)^2}\left\{-\left((p_1^2)^2+(p_2^2)^2+(p_3^2)^2\right)\frac{1}{\varepsilon}\right.$$
$$-2\left((p_1^2)^2+(p_2^2)^2+(p_3^2)^2\right) + 2\left(p_1^2p_2^2 + p_1^2p_3^2 + p_2^2p_3^2\right)$$
$$+\left(-(p_1^2)^2+(p_2^2)^2+(p_3^2)^2\right)\ln p_1^2 + \left((p_1^2)^2-(p_2^2)^2+(p_3^2)^2\right)\ln p_2^2$$
$$\left. + \left((p_1^2)^2+(p_2^2)^2-(p_3^2)^2\right)\ln p_3^2\right\} \qquad \text{(A3)}$$

where $n = 4-2\varepsilon$, and the factor A is defined by the formula (3.8). Here we omitted results which can be obtained from (A1)–(A3) by using the symmetry of the integrals (2.1) with respect to permutations of (ν_1, p_1), (ν_2, p_2) and (ν_3, p_3).

References

[1] Narison S 1982 *Phys. Rep.* **84** 263
[2] Eichten E, Hinchliffe I, Lane K and Quigg C 1984 *Rev. Mod. Phys.* **56** 579
[3] Fujimoto J, Igarashi M, Nakazawa N, Shimizu Y and Tobimatsu K 1990 *Prog. Theor. Phys. Suppl.* **100** 1
[4] Smirnov V A 1991 *Renormalization and Asymptotic Expansions* (Basel: Birkhäuser)
[5] Arbuzov B A 1988 *Particles and Nuclei* **19** 5
[6] Davydychev A I 1991 *Phys. Lett.* **263B** 107
[7] Tkachov F V 1981 *Phys. Lett.* **100B** 65
Chetyrkin K G and Tkachov F V 1981 *Nucl. Phys.* B **192** 159
[8] 'tHooft G and Veltman M 1972 *Nucl. Phys.* B **44** 189
Bollini C G and Giambiagi J J 1972 *Nuovo Cimento* B **12** 20
[9] Leibbrandt G 1975 *Rev. Mod. Phys.* **47** 849
[10] Boos E E and Davydychev A I 1991 *Teor. Mat. Fiz.* **89** 56

[11] Bailey W N 1935 *Generalized Hypergeometric Series* (Cambridge: Cambridge University Press)
[12] Davydychev A I 1991 *J. Math. Phys.* **32** 1052; 1992 *J. Math. Phys.* **33** 358
[13] Boos E E and Davydychev A I 1987 *Vestn. Mosk. Univ.* Ser.3 **28** 8
[14] Erdelyi A, Magnus W, Oberhettinger F and Tricomi F G 1953 *Higher Transcendental Functions (Bateman manuscript project)* vol 1 (New York: McGraw-Hill)
[15] Ussyukina N I 1983 *Teor. Mat. Fiz.* **54** 124
Belokurov V V and Ussyukina N I 1983 *J. Phys. A: Math. Gen.* **16** 2811
Kazakov D I 1984 *Teor. Mat. Fiz.* **58** 345
Kazakov D I and Kotikov A V 1987 *Teor. Mat. Fiz.* **73** 348
[16] 'tHooft G and Veltman M 1979 *Nucl. Phys.* B **153** 365
[17] Prudnikov A P, Brychkov Yu A and Marichev O I 1986 *Integrals and Series. Additional Chapters* (Moscow: Nauka)
[18] van Neerven W L 1986 *Nucl. Phys.* B **268** 453
[19] Alekseev A I 1988 *Proc. II Workshop 'Infrared Behaviour in QCD'* (Tbilisi: Tbilisi University Press) p 232
[20] Kramer G and Lampe B 1987 *J. Math. Phys.* **28** 945
[21] Alekseev A I, Arbuzov B A and Baykov V A 1982 *Teor. Mat. Fiz.* **52** 187
Milgram M S and H C Lee 1985 *J. Comput. Phys.* **59** 331
[22] Hearn A C 1987 *REDUCE User's Manual (RAND publication CP78)* (Santa Monica: RAND)